positive (stretching of the lowest filament) reduced moment decreases with increasing v, while the numerical value of the maximum negative moment increases just as in the case of a semi-infinite beam discussed above.

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SHOCK WAVE PROPAGATION IN ELASTIC-PLASTIC MEDIA

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It is shown that neutral shock waves, on which the plastic deformations are continuous, and waves on which the plastic deformations are discontinuous, can exist in ideal and hardened elastic-plastic media. Conditions for the existence of waves of the second kind are written down, the velocities of all the mentioned waves are determined in ideally plastic bodies for arbitrary convexity of the flow and Tresca conditions, and in hardened bodies for kinematic and isotropic hardening. Relationships are obtained for the discontinuities upon passage through the wave surface.

The behavior of shock waves during propagation under Mises and Tresca flow conditions is investigated by using the kinematic second-order compatibility conditions. It is shown that the shock wave intensity varies according to laws of geometric optics.

Questions of shock wave propagation in elastic-plastic media have been examined in [1 - 3]. Relationships on the shock waves in hardened elastic-plastic bodies have been derived under the assumption that simple loading occurs on the shock [1]. Reltionships on shock waves in plane ideally elastic-plastic bodies have been obtained in [2] by using the theory of generalized functions. The results of [1, 2] do not agree. The relationships proposed in [1] impose a lesser quantity of constraints on the shock wave parameters as compared with the results obtained in [2]. These relationships were utilized in [3] in solving the problem of oblique impact on an ideal elastic-plastic half-space.

Fundamental relationships on shock waves in elastic-plastic bodies are obtained below from thermodynamic considerations.

1. Let us consider an ideal elastic-plastic material. The total deformations consists of elastic and plastic parts

$$e_{ij} = e_{ij}^{e} + e_{ij}^{p} = (u_{i,j} + u_{j,i}) / 2 \tag{1.1}$$

The elastic deformations are connected with the stresses by means of Hooke's law

$$\sigma_{ij} = \lambda e_{kk}^{e}\delta_{ij} + 2\mu e_{ij}^{e} \tag{1.2}$$

In the plastic domain the stresses satisfy the plasticity condition

$$f(\sigma_{ij}) = k \tag{1.3}$$

The flow surface (1.3) is assumed nonconcave in the stress space and independent of the first invariant of the tensor σ_{ij}. The plastic strain rates are connected with the stresses by means of the associated flow law

$$\varepsilon_{ij}^{p} = \psi\partial f / \partial\sigma_{ij} = \psi f_{ij} \tag{1.4}$$

The relation between the stresses and strain rates can be represented as

$$S_{ij} = \sigma_{ij} - {}^{1}/_{3}\,\sigma_{kk}\delta_{ij} = \partial D / \partial\varepsilon_{ij}^{p} \tag{1.5}$$

where $D\ (\varepsilon_{ij}^{p})$ is the dissipation function. The relationships (1.5) and (1.4) are equivalent [4], and D is a homogeneous function of first degree in ε_{ij}.

The relationships (1.1) - (1.3) define the connection between the states of stress and strain in an elastic-plastic body in domains where the stresses and rates of displacement are continuous. If these parameters undergo abrupt changes in some domain, then it is necessary to rely upon thermodynamic considerations in the analysis.

Following the ideas elucidated in [5, 6], the heat flow equation and the equation of the second law of thermodynamics are written as

$$dU = \sigma_{ij}de_{ij} / \rho + dq^{e}, \qquad dF = dU - d(TS)$$
$$TdS = dq^{e} + dq', \qquad dq' = \tau_{ij}de_{ij}^{p} / \rho \geqslant 0 \tag{1.6}$$

Here U is the internal energy, F is the free energy, T is the absolute temperature, S is the entropy, ρ is the density, dq^{e} is the external heat influx, τ_{ij} are the components of some tensor characterizing energy dissipation. For an ideally plastic body $\tau_{ij} = \sigma_{ij}$. Henceforth, the strains are assumed small, and the Lagrange representation is utilized.

Let us consider a shock wave in an elastic-plastic material (a shock wave is understood to be a surface being propagated in space on which the displacements are continuous but the velocities and stresses undergo discontinuities). The "plus" and "minus" superscripts henceforth denote values of the quantities ahead of and behind the wave front, respectively.

Let us write the relationship on surfaces of strong discontinuity which has been obtained in [5]. The mass conservation condition

$$\rho^{+}c^{+} = \rho^{-}c^{-} \tag{1.7}$$

should be satisfied on the shock. Since $\rho = \rho_0 (1 + e_{kk})$, then because of the smallness of the strains on the shock $\rho^+ = \rho^- = \rho_0$. It follows from (1.7) that $c^+ = c^- = c^0$, i.e., we neglect the change in velocity of wave motion upon passage through the front of the wave surface.

Now the condition of momentum conservation can be represented as

$$[\sigma_{ij}] \nu_j + \rho c [\nu_i] = 0, \qquad [v_i] = v_i^+ - v_i^- \tag{1.8}$$

Here and henceforth, the square brackets denote the difference between the appropriate quantities on both sides of the surface of discontinuity, v_i is the velocity of material point motion, and ν_i is the unit vector of the normal to the surface of discontinuity.

Following [5], we write the law of energy conservation as

$$[\sigma_{ij} v_i] \nu_j - \rho (1/2 [v_i v_i] + [U]) c - [q_n] = 0 \tag{1.9}$$

where q_n^+ and q_n^- is the external influx of additional specific energy through the surface of discontinuity. Neglecting effects connected with heat exchange, it is possible to put $[q_n] = 0$ as well as $dq^e = 0$ in the relationships (1.6) for elastic-plastic bodies.

Let us compute the magnitude of the internal energy jump from (1.6) by assuming that the relationships (1.1) - (1.5) hold upon passage from the state e_{ij}^{p+} to the state e_{ij}^{p-}. This will hold if the rheological model of the body does not change within the transition layer simulating the shock wave. Separating the total strain in (1.6) into elastic and plastic for $dq^e = 0$, we obtain

$$\rho dU = \sigma_{ij} de_{ij}^e + \sigma_{ij} de_{ij}^p = 1/2 d\sigma_{ij} e_{ij}^e + \sigma_{ij} de_{ij}^p$$

Hence

$$\rho [U] = \frac{1}{2} [\sigma_{ij} e_{ij}^e] + A, \qquad A = \int_{e_1}^{e_2} \sigma_{ij} de_{ij}^p \tag{1.10}$$

$$e_1 = e_{ij}^{p-}, \qquad e_2 = e_{ij}^{p+}$$

From the second law of thermodynamics there follows that $A \leqslant 0$.

Let us note that the jump in internal energy is determined from (1.10) if the integral in the right side of (1.10), which depends on an unknown path of integration in the plastic strain space, is known.

From the Cauchy formula and the kinematic and geometric compatibility conditions for the jumps in total strains, we have

$$[e_{ij}] = [e_{ij}^e] + [e_{ij}^p] = - 1/2 ([v_i] \nu_j + [v_j] \nu_i) c^{-1} \tag{1.11}$$

From (1.2) there follows

$$[\sigma_{ij}] = \lambda [e_{kk}^e] \delta_{ij} + 2\mu [e_{ij}^e] \tag{1.12}$$

Utilizing (1.8), (1.10) - (1.12) and the reciprocity relationship $(\sigma_{ij}^+ e_{ij}^{e-} = \sigma_{ij}^- e_{ij}^{e+})$ the energy conservation equation (1.9) can be represented as

$$- 1/2 (S_{ij}^+ + S_{ij}^-) [e_{ij}^p] + A = 0 \tag{1.13}$$

Let us extract the path for which the functional A takes the maximum value, out of all possible paths connecting the points e_{ij}^{p+} and e_{ij}^{p-}. The Euler variational equation for the functional A is

$$\frac{d}{dz} \frac{\partial D}{\partial e_{ij}^p} = 0 \tag{1.14}$$

where z is the arclength of the trajectory $e_{ij}^p(z)$ in the plastic strain space. It follows from (1.5) and (1.14) that the maximum value of the functional A is achieved on the path for which S_{ij} = const, and we obtain from (1.10)

$$\max A = S_{ij}[e_{ij}^p] \tag{1.15}$$

Let us compute the maximum of the quantity $B = -\frac{1}{2}(S_{ij}^+ + S_{ij}^-)[e_{ij}^p]$ for fixed values of $[e_{ij}^p]$ and under the conditions that S_{ij}^+ and S_{ij}^- do not exceed the yield point

$$f(\sigma_{ij}^+) \leqslant k, \qquad f(\sigma_{ij}^-) \leqslant k$$

By determining the conditional extremum B we obtain that σ_{ij}^+ and σ_{ij}^- should satisfy the relationships

$$[e_{ij}^p] = \Psi_1 \partial f / \partial\sigma_{ij} - = \Psi_2 \partial f / d\sigma_{ij}^+ \tag{1.16}$$

where Ψ_1, Ψ_2 are undetermined Lagrange multipliers.

If the plasticity condition (1.3) is convex, then it follows from (1.16) that $S_{ij}^+ = S_{ij}^- = S_{ij}$, and

$$\max B = -S_{ij}[e_{ij}^p] \tag{1.17}$$

Since the strain rates are proportional to $[e_{ij}^p]$ on the path for which the functional A takes on the maximum value, then the values of S_{ij} in (1.15) and the values of S_{ij} determined from (1.16) should agree. This latter results from the equivalence relationships of theories constructed by using the plastic potential and the associated flow law [4].

Since $\max A = -\max B$, the equality (1.13) can only hold under the conditions that A and B take on the maximum values simultaneously or $[e_{ij}^p] = 0$. Therefore if $[q_n] = 0$ and $[e_{ij}^p] \neq 0$, then (1.9) is satisfied only under the condition that the stresses on both sides of the surface of discontinuity satisfy (1.16).

If a plastically incompressible body has a convex flow surface, then it follows from (1.16) that $[S_{ij}] = 0$ on the surface of discontinuity. If the plasticity condition of an isotropic body has a nonconcave portion, then (1.16) may also be satisfied even under the condition that $[e_{ij}^p]$ is orthogonal to the flat portion of the flow surface. Then it follows from the isotropy of the tensor relationship (1.16) that the direction cosines of the principal axes of the tensors $[e_{ij}^p]$, S_{ij}^+ and S_{ij}^- agree; the principal values of S_i^+ and S_i^- can differ, however $S_i^+[e_i^p] = S_i^-[e_i^p]$ since $[S_i]$ and $[e_i\ ]$ are orthogonal.

It follows from the above that shock waves of two kinds are possible in ideal elastic-plastic bodies; neutral waves and waves on which the plastic strains undergo a discontinuity. In the latter case, the stress deviators are continuous on the surface of discontinuity if the plasticity condition is convex. For nonconcave plasticity conditions, the direction cosines of the principla axes of the stress tensor are continuous on the surface of discontinuity. The jumps in the plastic strains are connected with the stresses by means of the relationships (1.16).

2. Let us examine the corollary of the results obtained above for media satisfying the Mises and Tresca platicity condition.

If the plastic strains are continuous on the shock, then

$$[e_{ij}^p] = 0, \quad [e_{ij}^e] = [e_{ij}] = -([v_i]\nu_j + [v_j]\nu_i)/2c$$

There hence results from (1.12)

$$-c\,[\sigma_{ij}] = \lambda\,[v_k]\,\nu_k\delta_{ij} + \mu\,([v_i]\,\nu_j + [v_j]\,\nu_i) \tag{2.1}$$

The relationship (2.1) should be considered jointly with (1.8). Equations (1.8), (2.1) agree identically with the relationships (1.8), (1.9) in [7], hence let us limit ourselves to presenting the final result.

Irrotational and equivoluminal neutral shock waves can exist in an elastic-plastic body. An irrotational wave is propagated with the velocity $c = \sqrt{(\lambda + 2\mu) / \rho}$, where

$$[v_i] = \omega\nu_i, \qquad c\,[\sigma_{ij}] = -\,\omega\,(\lambda\delta_{ij} + 2\mu\nu_i\nu_j) \tag{2.2}$$

Here ω is a quantity characterizing the wave intensity. An equivoluminal wave is propagated with the velocity $c = \sqrt{\mu / \rho}$, where

$$[v_i]\,\nu_i = 0, \qquad c\,[\sigma_{ij}] = -\,\mu\,([v_i]\,\nu_j + [v_j]\,\nu_i) \tag{2.3}$$

In contrast to elastic shock waves, the plasticity condition imposes a constraint on the quantities ω and $[v_i]$. Let us examine materials satisfying the Mises condition

$$S_{ij}S_{ij} = 2K^2 \tag{2.4}$$

From (2.2), (2.3) we obtain, respectively

$$cS_{ij}^- = cS_{ij}^+ + 2\mu\omega\,(\nu_i\nu_j - {}^1/_3\,\delta_{ij}) \tag{2.5}$$
$$cS_{ij}^- = cS_{ij}^+ + \mu\,([v_i]\,\nu_j + [v_j]\,\nu_i)$$

The loadings S_{ij}^- on the wave satisfy condition (2.4), from which

$$\mu\omega = {}^3/_4 c\left(-\,S_{ij}^+\nu_i\nu_j \pm \sqrt{S_{ij}^+\nu_i\nu_j + \frac{2}{3}\,(2k^2 - S_{ij}^+S_{ij}^+)}\right) \tag{2.6}$$

$$2\mu^2\,[v_i]\,[v_i] + 4\mu c S_{ij}^+\,[v_i]\,\nu_j - 2k^2 + S_{ij}^+S_{ij}^+ = 0 \tag{2.7}$$

The relationship (2.7) imposes constraints on the possible direction of $[v_i]$.

Let us note that, as has been shown in [1, 8, 9], the maximum velocity of loading wave propagation in elastic-plastic media equals $\sqrt{(\lambda + 2\mu) / \rho}$, hence, the perturbations behind the front of an irrotational shock wave, and the shock front itself cannot exert an influence on the state ahead of the wave front, and the quantities S_{ij}^+ are determined from the solution of the elastic problem. The intensity of the loading shock is then determined by (2.6), and the displacement velocities and stresses behind the shock front are determined from (2.2).

Upon considering the unloading waves $S_{ij}^-S_{ij}^- \leqslant 2k^2$, from which

$$c\omega S_{ij}^+\nu_i\nu_j + {}^2/_3\,\mu\omega^2 \leqslant 0, \qquad 2cS_{ij}^+\,[v_i]\,\nu_j + \mu\,[v_k]\,[v_k] \leqslant 0 \tag{2.8}$$

The connection between the shock wave parameters and the plastic strain rate jumps follows from the second order kinematic compatibility conditions. To obtain this connection is no different than from the corresponding reasoning in [7].

For irrotational and equivoluminal waves this connection is written as (the relationships (3.8), (3.11) in [7])

$$\rho c\,\frac{\delta\omega}{\delta t} = (\lambda + 2\mu)\,\Omega\omega + \mu\,[\varepsilon_{ij}^p]\,\nu_i\nu_j \tag{2.9}$$

$$\frac{\delta\,[v_i]}{\delta t} = c\Omega\,[v_i] + c\,([\varepsilon_{ij}^p]\,\nu_j - [\varepsilon_{mn}^p]\,\nu_m\nu_n\nu_i) \tag{2.10}$$

where Ω is the mean curvature of the wave surface. For the loading wave $[\varepsilon_{ij}^p] = -\,\varepsilon_{ij}^{p-}$

it follows from the associated flow law that $\varepsilon_{ij}^{p-} = \psi\partial f / \partial\sigma_{ji}^{-}$, while (2.9) is used to determine the quantity ψ for irrotational loading waves since ω, S_{ij}^{-} are known on the wave surface. In the case of equivoluminal loading waves, we have the differential equations (2.10), which should be considered jointly with the associated law and (2.7) under the assumptio that the S_{ij}^{+} are given, for the determination of the parameters $[v_i]$ and ψ behind the surface of discontinuity. On the unloading waves $[\varepsilon_{ij}^{p}] = \varepsilon_{ij}^{p+}$ and (2.9), (2.10) should be considered as differential equations to determine ω and $[v_i]$ respectively.

Let us examine shock waves on which plastic deformation occurs. Let us assume that the state of stress in the body corresponds to the yield surface of the Tresca plasticity condition

$$\sigma_1 - \sigma_2 = 2k \tag{2.11}$$

The stress tensor components σ_{ij} are connected to the principal stresses by the relationships

$$\sigma_{ij} = \sigma_1 l_i l_j + \sigma_2 m_i m_j + \sigma_3 n_i n_j \tag{2.12}$$

where l_i, m_i, n_i are the direction cosines of the principal directions of the tensor σ_{ij}, where

$$l_i m_i = l_i n_i = m_i n_i = 0, \quad l_i l_i = m_i m_i = n_i n_i = 1 \tag{2.13}$$

It follows from (1.8), (1.12), (2.11) and the continuity of the direction cosines of the principal directions of the tensors S_{ij}^{+}, S_{ij}^{-}, $[e_{ij}^{p}]$

$$\begin{gathered} c\,[\sigma_1]\,(\delta_{ij} - n_i n_j) + c\,[\sigma_3]\,n_i n_j = -\lambda\,[v_k]\,\nu_k\delta_{ij} - \\ -\mu\,\{[v_i]\,\nu_j + [v_j]\,\nu_i - \Psi\,(l_i l_j - m_i m_j)\} \\ [\sigma_1]\,(\nu_i - n_i\cos\varphi) + [\sigma_3]\,n_i\cos\varphi + \rho c\,[v_i] = 0, \quad \cos\varphi = n_i\nu_i \end{gathered} \tag{2.14}$$

Let us select a coordinate system x_i coincident with the principal axes $l_1 = m_2 = n_3 = 1$. Equations (2.14) are converted to

$$[v_i]\,\nu_j + [v_j]\,\nu_i = 0 \qquad (i \neq j) \tag{2.15}$$

$$\begin{gathered} c\,[\sigma_i] + \lambda\,[v_k]\,\nu_k + 2\mu\,[v_i]\,\nu_i + (\delta_{i1} - \delta_{i2})\,\mu\Psi = 0 \\ \sigma_i\nu_i = -\rho c\,[v_i] \qquad (\text{not summed over } i\ ) \end{gathered} \tag{2.16}$$

In order for the homogeneous system (2.15) to have nontrivial solutions, its determinant should vanish, hence $\nu_1\nu_2\nu_3 = 0$ Let us assume that $\nu_1 = 0$, $\nu_2 \neq 0$, $\nu_3 \neq 0$. It follows from (2.15) that $[v_1] = 0$. From (2.13), (2.16) we obtain

$$\rho c^2 = \frac{3\varkappa + 2}{4\varkappa + 3}\mu, \qquad \cos^2\varphi = \nu_3^2 = \frac{2\varkappa + 1}{4\varkappa + 3}; \qquad \varkappa = \frac{\lambda}{\mu} \tag{2.17}$$

Let us assume that $\nu_3 = 0$. It follows from (2.13), (2.15), (2.16) that

$$\rho c^2 = (\varkappa + 1)\,\mu, \ \nu_2 = [v_2] = 0 \ (\text{or} \quad \nu_1 = [v_1] = 0) \tag{2.18}$$

It is easy to see that the plastic deformations are continuous in the remaining cases. Substituting (2.17), (2.18) into (2.14), we obtain for the first and second waves, respectively

$$\begin{gathered} c\,[\sigma_1] = -c\,[\sigma_3] = -(3\lambda + 2\mu)\,\omega \\ [v_i] = \{-2\sqrt{(2\varkappa + 1)(4\varkappa + 3)}\,n_i + (4\varkappa + 3)\,\nu_i\}\,\omega \end{gathered} \tag{2.19}$$

$$c\,[\sigma_1] = -(\lambda + \mu)\,\omega, \qquad c\,[\sigma_3] = -\lambda\omega, \qquad [v_i] = \omega\nu_i \tag{2.20}$$

Let us examine the change in the quantity ω during wave propagation. From (1.1), (1.2), (1.4) follows

$$[\sigma_{ij,t}] = \lambda [v_{k,k}] \delta_{ij} + \mu ([v_{i,j}] + [v_{i,j}] - [\Psi f_{ij}]) \tag{2.21}$$

The equations of motion are

$$\sigma_{ij,j} = \rho v_{i,t} \tag{2.22}$$

Let us differentiate (2.11) and (2.13) with respect to time and the coordinates, and let us form the differences of the corresponding equations written on both sides of the wave surface. Utilizing these relationships and the second order compatibility condition, we obtain from (2.21), (2.22)

$$\begin{gathered}
-cM_1(\delta_{ij} - n_i n_j) - cM_3 n_i n_j - c([\sigma_1] - [\sigma_3] - \sigma_1^+ + \sigma_3^+)(a_i n_j + a_j n_i) + \\
+ 2kc(b_i m_j + b_j m_i) - \lambda L_k \nu_k \delta_{ij} - \mu (L_i \nu_j + L_j \nu_i) + 2\mu [\psi] (l_i l_j - m_i m_j) = \\
= A_{ij} = -(\delta_{ij} - n_i n_j)\delta[\delta_1]/\delta t - n_i n_j \delta[\sigma_3]/\delta t + ([\sigma_1] - [\sigma_3])(n_i^+ n_j^+)_{,t} + \\
+ \lambda g^{\alpha\beta} [v_k]_{,\alpha} x_{k\beta} \delta_{ij} + \mu g^{\alpha\beta} ([v_i]_{,\alpha} x_{j\beta} + [v_j]_{,\alpha} x_{i\beta})
\end{gathered} \tag{2.23}$$

$$\begin{gathered}
M_1(\nu_i - n_i \cos\varphi) + M_3 n_i \cos\varphi + ([\sigma_1] - [\sigma_3] - \sigma_1^+ + \\
+ \sigma_3^+)(a_i \cos\varphi + a_k \nu_k n_i) - 2k(b_i m_k \nu_k + m_i b_k \nu_k) + \rho c L_i = \\
= B_i = \rho\delta[v_i]/\delta t - g^{\alpha\beta}[\sigma_1]_{,\alpha}(x_{i\beta} - n_i x_{k\beta} n_k) - \\
- g^{\alpha\beta}[\sigma_3]_{,\alpha} x_{i\beta} n_i n_j + ([\sigma_1] - [\sigma_3])(n_i^+ n_j^+)_{,j}
\end{gathered} \tag{2.24}$$

Here

$$M_1 = \left[\frac{d\sigma_1}{dn}\right], \qquad M_3 = \left[\frac{d\sigma_3}{dn}\right], \qquad L_i = \left[\frac{dv_i}{dn}\right]$$

$$a_i = \left[\frac{dn_i}{dn}\right], \qquad b_i = \left[\frac{dm_i}{dn}\right], \qquad a_k n_k = b_k m_k = 0$$

Let us multiply (2.23) by δ_{ij}, $n_i n_j$ and (2.24) by n_i, ν_i. Eliminating M_1, M_3, $L_k \nu_k$, $L_k n_k$, $a_k n_k$ from the obtained relationships, and taking (2.17) and (2.18) into account, we obtain for the first and second waves respectively

$$A_{kk} \sin^2\varphi - A_{ij} n_i n_j (1 + \cos^2\varphi) + 2c(B_k \nu_k - 2B_k \nu_k \cos\varphi) = 0$$

$$A_{kk} - A_{ij} n_i n_j + 2cB_k \nu_k = 0$$

By using (2.19), (2.20) these equations can be represented as

$$\delta\omega / \delta t = c\Omega\omega$$

Therefore, the intensity of both waves changes according to the laws of geometric optics.

Let us assume that a convex plasticity condition holds. It hence follows from (1.12)

$$c[\sigma_{ij}] = -\lambda [v_k] \nu_k \delta_{ij} - \mu ([v_i] \nu_j + [v_j] \nu_i + c[e_{ij}^p]) \tag{2.25}$$

Since $[S_{ij}] = 0$, then $[\sigma_{ij}] = {}^1/_3 [\sigma_{kk}] \delta_{ij}$, and (2.25), (1.7) become

$$^1/_3\, c[\sigma_{kk}] \delta_{ij} = -\lambda [v_k] \nu_k \delta_{ij} - \mu ([v_i] \nu_j + [v_j] \nu_i + c[e_{ij}^p]) \tag{2.26}$$

$$^1/_3 [\sigma_{kk}] \nu_i + \rho c [v_i] = 0 \tag{2.27}$$

Eliminating the quantities $[v_i]$ from (2.26) by using (2.27) we obtain

$$^1/_3\, \rho c^2 [\sigma_{kk}] \delta_{ij} = {}^1/_3\, \lambda [\sigma_{kk}] \delta_{ij} + {}^2/_3\, \mu [\sigma_{kk}] \nu_i \nu_j - \mu [e_{ij}^p] \tag{2.28}$$

Equating the subscripts i and j in (2.28), we obtain the wave propagation velocity as

$$\rho c^2 = \lambda + {}^2/_3\, \mu \tag{2.29}$$

For the Mises plasticity condition it follows from (2.4) and (1.6) that on the surface of discontinuity

$$S_{ij} = 2\,[\sigma_{kk}]\,(\delta_{ij} - 3\nu_i\nu_j)\,/\,9\,[\psi] \tag{2.30}$$

Therefore, the shocks on which a jump in plastic deformations occur can be propagated at the velocity (2.29) only when the state of stress (2.30) holds ahead of the wave front. Assuming the coordinate axes x_i to be selected such that the direction ν_i coincides with the x_3-axis and the directions x_1, x_2 lie on the wave surface, we obtain from (2.30)

$$S_{11} = S_{22} = -2S_{33} = {}^2/_9\,[\sigma_{kk}]\,[\psi]^{-1}, \qquad S_{12} = S_{23} = 0$$

Consequently the considered shock can exist only under the condition that the wave surface contains two principal directions of the stress tensor and the complete plasticity condition $\sigma_1 = \sigma_2 = \sigma_3 \pm 2\sqrt{2}k$ holds ahead of the wave front. Let us note that $[\sigma_{kk}] = 3\sqrt{2}k\,[\psi]$ follows from the plasticity condition.

An equation describing the change in the quantity ω, and therefore also in all the wave characteristics during propagation, can be obtained by using the second order kinematic and geometric compatibility conditions.

Omitting the tedious discussions carried out earlier in [7, 9, 10] and above for the Tresca yield surface, let us limit ourselves to presenting just the final result

$$\delta\omega\,/\,\delta t = c\Omega\omega \tag{2.31}$$

It follows from (2.31) that the intensity of the considered wave changes according to laws of geometric optics.

3. Let us examine a hardening elastic-plastic material whose loading surface is

$$(S_{ij} - \eta e_{ij}^p)(S_{ij} - \eta e_{ij}^p) = 2\,(k + r\varepsilon)^2 \tag{3.1}$$

$$\varepsilon = \int_0^t \sqrt{\varepsilon_{ij}^{\cdot p}\varepsilon_{ij}^{\cdot p}}\,dt; \qquad \eta,\ r = \text{const}$$

The loading surface (3.1) combines the kinematic hardening proposed in [11, 12] and isotropic hardening. It follows from the associated flow law that

$$\varepsilon_{ij}^{\cdot p} = \Psi^{\cdot}\,(S_{ij} - \eta e_{ij}^p) \tag{3.2}$$

From (3.1) we obtain for the quantity ψ

$$\Psi^{\cdot} = \sqrt{\varepsilon_{ij}^{\cdot p}\varepsilon_{ij}^{\cdot p}}\,/\sqrt{2}\,(k + r\varepsilon)$$

It then follows from (3.2) that

$$S_{ij} = \eta e_{ij}^p + \sqrt{2}\varepsilon_{ij}^{\cdot p}\,(k + r\varepsilon)\,(\varepsilon_{ij}^{\cdot p}\varepsilon_{ij}^{\cdot p})^{-1/2}$$

The dissipation function corresponding to the loading surface (3.1) is

$$D = \sqrt{2}\,(k + r\varepsilon)\sqrt{\varepsilon_{ij}^{\cdot p}\varepsilon_{ij}^{\cdot p}} + \eta e_{ij}^p e_{ij}^{\cdot p} \tag{3.3}$$

Let us consider the equality (1.13). Taking account of (3.3), we have for the quantities from (1.13)

$$A = \int_0^T D\,dt = \int_0^T \sqrt{2}\,(k + r\varepsilon)\sqrt{\varepsilon_{ij}^{\cdot p}\varepsilon_{ij}^{\cdot p}}\,dt + \int_0^T \eta e_{ij}^p\varepsilon_{ij}^{\cdot p}\,dt =$$

$$= \sqrt{2}\,[\varepsilon] - \frac{1}{2}\,r\sqrt{2}\,[\varepsilon^2] - \frac{1}{2}\,\eta\,[e_{ij}^p e_{ij}^p]$$

Let us compute the maximum of the quantity $-{}^1/_2\,(S_{ij}^+ + S_{ij}^-)\,[e_{ij}^p]$ under the condition that

$$(S_{ij}^{\pm} - \eta e_{ij}^{p\pm})(S_{ij}^{\pm} - \eta e_{ij}^{p\pm}) \leqslant 2(k + r\varepsilon^{\pm})^2 \tag{3.4}$$

The conditional extremum holds when the equality sign is achieved in (3.4) and for

$$S_{ij}^{\pm} = \eta e_{ij}^{p\pm} - \sqrt{2}\,[e_{ij}^p]\,(k + r\varepsilon^{\pm})([e_{ij}^p]\,[e_{ij}^p])^{-1/2} \tag{3.5}$$

The relationship (1.13) hence becomes

$$\sqrt{2}k\,(\sqrt{[e_{ij}^p]\,[e_{ij}^p]} + [\varepsilon]) + {}^1/_2\, r\sqrt{2}\{(\varepsilon^+ + \varepsilon^-)\,[e_{ij}^p]\,[e_{ij}^p] + [\varepsilon^2]\} = 0 \tag{3.6}$$

The quantity $[\varepsilon]$ is the length of the loading trajectory in plastic strain space, and its maximum value

$$[\varepsilon] = -\sqrt{[e_{ij}^p]\,[e_{ij}^p]} \tag{3.7}$$

satisfies (3.6).

It hence follows that shocks of two kinds can exist in hardening elastic-plastic bodies with a loading surface of the kind (3.1). These will be neutral waves on which the plastic strains are continuous, and waves on which (3.5) will hold, where $[\varepsilon]$ is determined according to (3.7). Taking account of (3.7), we obtain from (3.5)

$$[S_{ij}] = (\eta + \sqrt{2}r)\,[e_{ij}^p] \tag{3.8}$$

It follows from (3.8), (1.12) that the connection between the stress and strain jumps is

$$[\sigma_{ij}] = \lambda_1\,[e_{kk}]\,\delta_{ij} + 2\mu_1\,[e_{ij}] \tag{3.9}$$

$$\lambda_1 = \frac{3\lambda(\eta + \sqrt{2}r) + 2\mu\,(3\lambda + 2\mu)}{3(\eta + \sqrt{2}r) + 6\mu}, \qquad \mu_1 = \frac{2\mu\,(\eta + r\sqrt{2})}{\eta + r\sqrt{2} + 2\mu} \tag{3.10}$$

It follows from (3.9) that the velocities and discontinuities are the same for waves in a hardening elastic-plastic material as in an elastic material whose elastic moduli are computed by means of (3.10).

Irrotational waves are propagated at the velocity

$$c_1 = \sqrt{(\lambda_1 + 2\mu_1)/\rho}$$

and the relationships

$$[v_i] = \omega\nu_i, \quad c_1\,[\sigma_{ij}] = -\omega\,(\lambda_1\delta_{ij} + 2\mu_1\nu_i\nu_j) \tag{3.11}$$

are satisfied on these waves.

The equivoluminal waves are propagated at the velocity

$$[v_i]\,\nu_i = 0, \qquad c_2\,[\sigma_{ij}] = -\mu_1([v_i]\,\nu_j + [v_j]\,\nu_i) \tag{3.12}$$

and the relationships

$$c_2 = \sqrt{\mu_1/\rho}$$

hold thereon.

Utilizing (3.11) and (3.12), we obtain the jumps in plastic deformation from (3.8). On the irrotational and equivoluminal waves we obtain, respectively

$$[e_{ij}^p] = \frac{2\mu_1\omega}{\eta + \sqrt{2}r}\left(\frac{1}{3}\delta_{ij} - \nu_i\nu_j\right), \qquad [e_{ij}^p] = -\frac{\mu_1([v_i]\,\nu_j + [v_j]\,\nu_i)}{\eta + \sqrt{2}r} \tag{3.13}$$

Now, it follows from (3.5) that the state of stress

$$S_{ij} = \eta e_{ij}^p \pm (k + r\varepsilon)\sqrt{2}\,({}^1/_3\,\delta_{ij} - \nu_i\nu_j) \tag{3.14}$$

$$S_{ij} = \eta e_{ij}^p \pm ([v_i]\,\nu_j + [v_j]\,\nu_i)(k + r\varepsilon)([v_k]\,[v_k])^{-1/2}$$

should hold ahead of the front of irrotational and equivoluminal waves.

Therefore, exactly as in an ideally plastic medium, shocks in a hardening body with loading surface (3.1) can occur only in certain exceptional cases when the state of stress (3.14) holds ahead of the wave front. The intensity of these waves changes according to the laws of geometric optics.

Let us note that it has been assumed in [1] for a hardening body that the stress trajectory in the space σ_{ij} is a radial line upon passage through the surface of discontinuity.

It follows from (3.8), (3.12), (3.14) that this hypothesis holds only under the condition of coaxiality of the stress and strain tensors, i.e., in the general case the hypotheses of [1] will contradict the laws of thermodynamics for a hardening body with the loading surface (3.1).

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